Chapter 3
Control of Nonhyperbolic Dynamical Systems Through Center Manifold Control

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Abstract This chapter proposes a simple approach for the control of nonlinear dynamical systems with nonhyperbolic equilibrium points. Such equilibrium points are generally much more difficult to analyze dynamically, and correspondingly the control of nonlinear systems in the vicinity of such points can often become more difficult. The aim is to bring about asymptotically stable behavior of the controlled system in the vicinity of the nonhyperbolic equilibrium point. A new way to control such systems is proposed here through control of their local center manifolds. A simple and effective methodology for doing this is provided, and its advantages are illustrated through several examples.

Keywords Dynamical Systems · Center manifold control · Nonhyperbolic dynamical systems · Asymptotic stability · Examples

3.1 Introduction

In local analysis of nonlinear dynamical systems, one of the most useful and powerful results is the Grobman-Hartman (GH) result [1, 2] that proves topological conjugacy in the vicinity of a hyperbolic equilibrium (fixed) point between a nonlinear dynamical system and its linearization at that equilibrium point. By a hyperbolic equilibrium point is meant one at which the Jacobian of the system has eigenvalues whose real parts are nonzero, and by topological conjugacy is meant that there is a continuous invertible bijective mapping that preserves the direction of time and that maps the phase portrait of the nonlinear system to that of its linearization in the vicinity of the equilibrium point.

When an equilibrium point of a nonlinear system is not hyperbolic, the Grobman-Hartman (GH) result can no longer be used, and the dynamics that ensues in

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its vicinity can be considerably more complex and therefore more challenging to control. One of the approaches to understanding the consequent dynamical behavior in this situation is the use of results that utilize the concept of center manifolds [3–7]. The theory of center manifolds is a rigorous development of the theory of nonlinear differential equations that includes systems with multiple timescales, especially systems that have so-called slow and fast variables.

In this chapter, we consider an approach which is simple and straightforward for the control of such systems and obtain controls that ensure asymptotic stability at the nonhyperbolic equilibrium point along with a region of attraction that is often reasonable in "size" (in phase space) from a practical engineering viewpoint.

Consider the nonlinear dynamical system described by the equations:

$$\begin{aligned} \dot{x} &= Ax + f(x, y) \\ \dot{y} &= By + g(x, y) \end{aligned} \tag{3.1}$$

in which $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A and B are constant matrices and the dots denote differentiation with respect to time. The matrix A is assumed to have eigenvalues, λ_A, whose real parts are zero, and the matrix B is assumed to have eigenvalues, λ_B, whose real parts are negative. The matrix B is taken to be a stabile matrix because our interest is in stability/instability of the nonhyperbolic equilibrium points of systems. The functions $f(x, y)$ and $g(x, y)$ are assumed to be C^2 with $f(0, 0)) = 0$, $Df(0, 0) = 0$, $g(0, 0) = 0$, and $Dg(0, 0) = 0$, where $Df(0, 0)$ denotes the Jacobian of f evaluated at $(0, 0)$. The set $S \subset \mathbb{R}^{n+m}$ is a local invariant manifold of Eq. (3.1) if for any solution $(x(t), y(t))$ with $(x(0), y(0)) \subset S$ there is a positive time T such that $(x(t), y(t)) \subset S$ for $t \in [0, T]$.

One could also consider a more general system that is topologically conjugate to Eq. (3.1), described by $\dot{z} = w(z), z \in \mathbb{R}^{n+m}$ whose linearization about the equilibrium point $z = 0$ yields a Jacobian $Dw(0)$ that has n eigenvalues whose real parts are zero and m eigenvalues whose real parts are negative. However, in this chapter we will continue to use Eq. (3.1) since it is more explicit.

Were the functions f and g in Eq. (3.1) to be identically zero, it would simplify to $\dot{x} = Ax, \dot{y} = By$. Then $y = 0$ would be an invariant manifold, meaning that for $y(t = 0) = 0$, $y(t) = 0$, for $\forall t$, and the flow on this subspace would be given by the simpler equation $\dot{u} = Au$. Were the initial condition to be $y(t = 0) = \delta$, for "small" δ, $y(t)$ would exponentially go to zero, that is, to the invariant manifold $y = 0$; the long-term behavior of system (3.1) would again then be provided by the equation $\dot{u} = Au$. This simpler self-contained equation with $u \in \mathbb{R}^n$ can be viewed as a kind of "reduction" of the $k := n + m$ dimensional system (3.1) to one of lower dimension, namely, n.

The generalization of this idea of decoupling the dynamics and, in essence, reducing the order of the system's dimension when the functions f and g are not identically zero is provided by the central results obtained in center manifold theory. It can be shown that the system described by Eq. (3.1) has the following properties [2–7]:

Existence

There exists a smooth (C^k) m-dimensional center manifold of the form $y = h(x)$, $\|x\| < \delta$, such that $y(0) = h(0) = 0$ and $Dh(0) = 0$. The dynamics on this center manifold are governed by the equation:

$$\dot{u} = Au + f\,(u, h(u))\,. \tag{3.2}$$

In other words, system (3.1) whose dimension is $k := n + m$ possesses a lower-dimensional center manifold of dimension m, and the dynamics on this manifold is self-contained and given by the n-dimensional system (3.2).

Stability

Suppose that the zero solution of Eq. (3.2) is stable (asymptotically stable) (unstable), then the zero solution of Eq. (3.1) is stable (asymptotically stable) (unstable).

Asymptotic Behavior of Trajectories

Suppose that the zero solution is stable. Let $(x(t), y(t))$ be a solution of Eq. (3.1) with the initial condition $((x(0), y(0))$ sufficiently small (though in practice, sufficiently small may be quite substantial); then there exists a solution of Eq. (3.2), $u(t)$, such that as $t \to \infty$:

$$x(t) = u(t) + \mathrm{O}\left(\mathrm{e}^{-\gamma t}\right),\; y(t) = h\,(u(t)) + \mathrm{O}\left(\mathrm{e}^{-\gamma t}\right), \tag{3.3}$$

where $\gamma > 0$ is some constant.

In other words, from a sizable range of initial conditions near the origin, all solutions of Eq. (3.1) tend exponentially in time to a solution on the center manifold. The n-dimensional reduced system given by Eq. (3.2) on the m-dimensional center manifold $y = h(x)$ faithfully models the original system (3.1) as $t \to \infty$.

Approximation

To get an approximation of the center manifold, $y = h(x)$ is substituted in the second relation is Eq. (3.1) to obtain, using the chain rule,

$$Dh(x)\,[Ax + f\,(x, h(x))] - Bh(x) - g\,(x, h(x)) = 0 \tag{3.4}$$

which is required to be solved along with the conditions that $h(0) = 0$ and $Dh(0) = 0$. An approximation to $h(x)$ is obtained by defining:

$$M\tilde{h}(x) := D\tilde{h}(x)\left[Ax + f\left(x, \tilde{h}(x)\right)\right] - b\tilde{h} - g\left(x, \tilde{h}(x)\right) \tag{3.5}$$

and attempting to solve approximately $M\tilde{h} = 0$. Here, let $\tilde{h}$ be a map from $\mathbb{R}^n$ into $\mathbb{R}^m$ with $\tilde{h}(0) = 0$ and $D\tilde{h}(0) = 0$. If as $x \to 0$, $M\tilde{h} = \mathrm{O}\left(\|x\|^q\right)$ then $h(x) = \tilde{h}(x) + \mathrm{O}\left(\|x\|^q\right)$. In other words, if Eq. (3.4) is satisfied to some order of accuracy, then the center manifold will have been found to the same order of accuracy.

Often, in bifurcation studies, a p-vector, ε, consisting of constant parameters, is present in Eq. (3.1). Then, if [1]:

$$M\tilde{h} = \mathrm{O}\left(\|x\|^q, \|\varepsilon\|^s\right), \text{ then } h(x) = \tilde{h}(x) + \mathrm{O}\left(\|x\|^q, \|\varepsilon\|^s\right). \tag{3.6}$$

Remark 1 The center manifold, $y = h(x)$, is generated by our desire that it satisfies Eq. (3.4), but finding a suitable $h(x)$ so that this equation is exactly satisfied is usually not possible, because it would be tantamount to solving the nonlinear Eq. (3.1). Hence the approximate solution correct to some order is sought by setting $M\tilde{h} = 0$. Often a power series approximation is used.

Assuming now that Eq. (3.1) is a model of a naturally occurring system or an engineered system, we could envisage controlling the system by controlling its center manifold so in the presence of the control: (1) instead of getting an approximation to $h(x)$ that is correct to some $\mathrm{O}(\|x\|^q, \|\varepsilon\|^s)$ as in Eq. (3.6), we can obtain the function exactly, and (2) we can make a system that is unstable at a hyperbolic equilibrium point asymptotically stable by suitable choices of center manifolds that are cognizant of practical needs, limitations, and requirement, for controlling the system. It is these aspects of control that are related to center manifold theory that this chapter is centrally concerned with.

3.2 Control of Dynamical Systems with Nonhyperbolic Equilibrium Points

Our intent is to provide a method to control the unstable dynamics at a nonhyperbolic equilibrium point (which is taken, with no loss of generality, to occur at the origin) and make this nonhyperbolic equilibrium point asymptotically stable.

To illustrate the central ideas addressed in this chapter, we begin with a simple example. Consider the system of equations:

$$\begin{aligned} \dot{x} &= xy + ax^3 + bxy^2 = 0x + f(x, y) \\ \dot{y} &= -y + cx^2 + dx^2 y = -y + g(x, y) \end{aligned} \tag{3.7}$$

in which a, b, c, and $d \in \mathbb{R}$ are constants and form the parameter 4-vector $\varepsilon = [a, b, c, d]^{\mathrm{T}}$. The matrix $A = 0$, and $B = -1$ here. By the Existence property above, a center manifold $y = h(x)$ exists, which can be obtained by solving Eq. (3.4), which here is:

$$h'(x)\left[xh(x) + ax^2 + bxh(x)^2\right] + y - cx^2 - dx^2 y = 0. \tag{3.8}$$

Using Eq. (3.6), an approximation to the center manifold is obtained by setting $M\tilde{h}(x) = 0$. Assuming that $M\tilde{h}(x) = O\left(x^2\right)$ this gives (see [1] for details):

$$M\tilde{h}(x) = \tilde{h}(x) - cx^2 + O\left(x^4\right) = 0, \tag{3.9}$$

so that

$$h(x) = \tilde{h}(x) + O\left(x^4\right) = cx^2 + O\left(x^4\right). \tag{3.10}$$

Eq. (3.2) then yields, from the Existence property given in Sect. 3.1 above, that

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (a + c)\,u^3 + O\left(x^5\right). \tag{3.11}$$

When $a + c < 0$, Eq. (3.11) is asymptotically stable at the origin, and by the Stability property of the previous section, so is Eq. (3.7). When $a + c > 0$, Eq. (3.11) is unstable at the origin, and therefore so is Eq. (3.7). The situation when $a + c = 0$ poses a difficulty and a higher order approximation is required [1].

While this might be satisfactory in certain situations, from an engineering controls perspective, it leaves open the following practical questions:

1. What if the parameters a and c describing the system were such that $a + c > 0$? How would one control such a system with a nonhyperbolic equilibrium point and make the nonhyperbolic equilibrium point asymptotically stable?
2. Often one is interested in controlling a system to follow a given trajectory or in controlling a system so that *all* its orbits in phase space lie on a prespecified curve (surface, in higher dimensions) as they approach an equilibrium point, so how does one control the system?
3. For engineering applications, one is often interested in the exact description of a local center manifold, instead of obtaining a center manifold that, in this example, is locally approximated by $h(x) = cx^2 + O(x^4)$. How can one ensure that the center manifold is exactly given by $h(x) = cx^2$?
4. More generally, what if one wanted to create, as we shall see below, a center manifold for a given system described by $y = s(x)$, where s is a preferred function of x?

It is the answers to questions like these, which arise mainly from a "controls" perspective, that this chapter deals with.

We note that: (i) Eq. (3.10) gives only an approximation to the center manifold and (ii) Eq. (3.11) that gives the dynamics on the center manifold is also approximately and qualitatively obtained, with our understanding pinned to what happens as $\|x\| \to 0$, though it might be applicable to a region considerably larger.

From an engineering viewpoint, we may want that the system has a desired center manifold and that the dynamics on it evolve in a definite manner. Noting that the center manifold relies on having $y = h(x)$ satisfy the second equation of system (3.7), from an engineering viewpoint, one can add a control to this equation so that we now have the controlled system:

$$\begin{aligned} \dot{x} &= xy + ax^3 + bxy^2 &&= 0x + f\,(x, y) \\ \dot{y} &= -y + cx^2 + dx^2y + w(x) &&= -y + g_{\mathrm{c}}\,(x, y)) \end{aligned} \tag{3.12}$$

where $w(x)$ is the nonlinear control that is added, with $w(0) = \dot{w}(0) = 0$; the subscript "c" on g denotes the controlled system.

One could perhaps then determine the control $w(x)$ and, instead of having a somewhat vaguely defined approximation of the center manifold given in Eq. (3.10), demand that the center manifold of the controlled system be exactly $y = h(x) = c_0 x^\beta$ for a suitable (integer) value of β. We note that since we require $h(0) = Dh(0) = 0$, we must have $\beta \geq 2$. Substituting $y = c_0 x^\beta$ in the second equation in (3.12), we get

$$\begin{aligned} w(x) &= \beta c_0 x^{\beta-1}\left[c_0 x^{\beta+1} + ax^3 + bc_0^2 x^{2\beta+1}\right] + c_0 x^\beta - cx^2 - c_0 dx^{\beta+2} \\ &= c_0^2 \beta x^{2\beta}\left(1 + bc_0 x^\beta\right) + c_0\,(a\beta - d)\,x^{\beta+2} + c_0 x^\beta - cx^2. \end{aligned} \tag{3.13}$$

This provides the explicit control $w(x)$ to be applied to Eq. (3.7) so that the center manifold of the controlled system is exactly given by $y = c_0 x^\beta$, and the dynamics unfurls on this center manifold according to the equation:

$$\dot{u} = uh(u) + au^3 + buh^2(u) = c_0 u^{\beta+1} + au^3 + c_0^2 bu^{2\beta+1}. \tag{3.14}$$

Notice that the dynamics only depends on the parameters a and b of the uncontrolled system that appear in the (x-equation) and the parameter c_0 that defines the "controlled" center manifold. By the Stability property in the previous section, asymptotic stability of the origin in system (3.12) is assured when the origin in Eq. (3.14) is asymptotically stable.

From a controls perspective, in order to ensure stability of the nonhyperbolic origin, the "controlled" center manifold needs to be selected with an eye to making the reduced system, which is described by Eq. (3.2), stable at the origin. As seen from Eq. (3.14), β would thus need to be an even integer.

Numerical Example 1 The following numerical example captures the usefulness of this simple approach to the control of nonlinear systems in the vicinity of unstable nonhyperbolic equilibrium points rendering them asymptotically stable.

(i) When $a + c < 0$, as suggested by Eq. (3.10), a suitable controlled center manifold can be taken as $y = c_0 x^\beta$ with $c_0 = c = -2$ and $\beta = 2$. This is brought about by controlling the second equation in (3.12) with the control $w(x)$ given in Eq. (3.13)

The phase plot of the controlled system that employs the control described in Eq. (3.13) with the parameter vector $\varepsilon = [0.5, 1, -2, 1]^{\mathrm{T}}$ is shown in Fig. 3.1. As expected, the equilibrium point at the origin is seen to be asymptotically stable since $a + c_0 = -1.5$. The thin solid line, convex upward, shows the manifold which has exactly the equation $y = cx^2$, and the local dynamics on this manifold is described using Eq. (3.14) by the equation

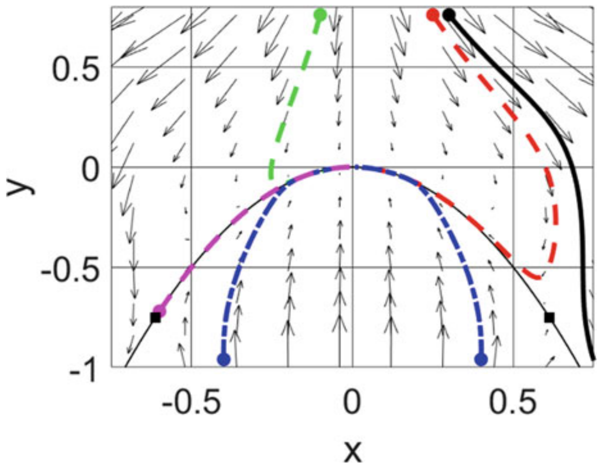


Fig. 3.1 System 3.12 is controlled so that its center manifold is $y = -2x^2$. $a = 0.5$, $b = 1$, $c = -2$, $d = 1$

$$\dot{u} = -1.5u^3 + 4u^5.$$

which shows that the equilibrium point $u = 0$ is asymptotically stable for orbits for which $|u|$ is sufficiently small. The control $w(x)$ provided in Eq. (3.12) is

$$w(x) = 8x^4\left(1 - 2x^2\right).$$

Orbits starting from different initial conditions are shown by dash lines. The initial conditions for each orbit are shown by a solid circle. As seen, the origin is asymptotically stable over a generous region of initial conditions around it. Trajectories move toward the center manifold and move along it to reach the origin.

However, the plotted vector field and the orbit shown by the solid line show that the origin is not globally asymptotically stable; in fact, a pair of unstable equilibrium points exit at $(\pm 0.6123, -0.75)$ shown by solid squares in the figure showing that the Stability property, though it yields asymptotic stability of the equilibrium point, is, of course, local. Determining the basin of attraction of the nonhyperbolic equilibrium point is generally difficult, and it depends both on the parameters of the system and the choice of the desired center manifold.

Remark 2 Where the parameter $a < 0$, one could also have controlled the stability of the nonhyperbolic equilibrium point at the origin and the behavior of nearby orbits by controlling the center manifold of the controlled system to be $y = h(x) = c_0 x^4$, $c_0 = \pm 1$, through the use of a suitable control $w(x)$ (as given in Eq. (3.13)). For then, Eq. (3.14) would become:

$$\dot{u} = uh(u) + au^3 + buh^2(u) = au^3 \pm u^5 + bu^9, a < 0$$

whose origin is asymptotically stable for $|u|$ sufficiently small.

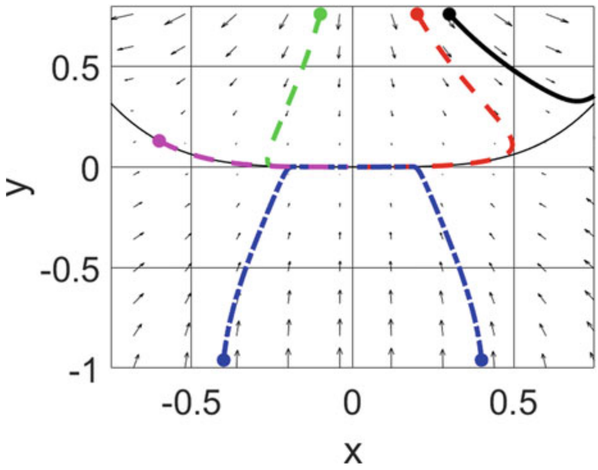


Fig. 3.2 System (3.12) is controlled so that its center manifold is $y = x^4$. $a = -0.5, b = 1, c = -2, d = 1$

This is illustrated for the parameter vector $\varepsilon = [-0.5, 1, -2, 1]^T$, now with $a = -0.5$ and $c_0 = 1$ Using the control $w(x) = 4x^8(1 + x^4) + 2x^2 + x^4 - 3x^6$ obtained from Eq. (3.13), Fig. 3.2 shows the phase portrait and the vector field of the controlled system. The center manifold is now controlled to lie on the curve $y = x^4$ shown by the thin line, which is concave upward now. The equilibrium point at the origin is asymptotically stable. Representative orbits that start in the vicinity of the origin move toward it exponentially fast. (Note the orbit shown by the solid line that does *not* go the origin, illustrating the local nature of the stability.)

In the vicinity of the origin, the system closely follows the trajectory given by the equation $y = x^4$, and the local dynamics (Eq. (3.14)) on this manifold is given by $\dot{u} = -0.5u^3 + u^5 + u^9$.

(ii) When $a + c > 0$, using the Stability property of the previous section, we see from Eq. (3.11) that the uncontrolled system (3.7) is unstable at the origin. However, asymptotic stability of the origin can still be achieved by controlling the center manifold of system through the addition of a control $w(x)$ as in Eq. (3.12). For $\beta = 2$, the center manifold of the controlled system becomes $y = c_0x^2$. With this as the controlled center manifold, by using Eq. (3.14) with $\beta = 2$, we see that asymptotic stability of the origin of the controlled system can then be guaranteed by choosing $c_0 < -a$.

Numerical Example 2 Consider the controlled system with the parameter vector $\varepsilon = [0.5, 1, 0.5, 1]^T$ for which $a + c = 1 > 0$. The phase portrait of the uncontrolled system (3.7) in the vicinity of its unstable nonhyperbolic equilibrium point at the origin is shown in Fig. 3.3a. As expected from Eq. (3.11), this nonhyperbolic equilibrium point is unstable.

We now use the control:

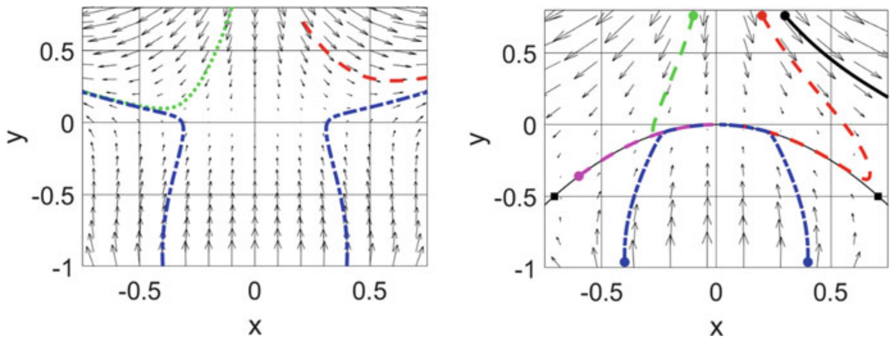


Fig. 3.3 (**a**) Unstable nonhyperbolic equilibrium point of system (Eq. (3.7)) when $a + c > 0$ of uncontrolled system. (**b**) Exponentially stable nonhyperbolic equilibrium point of system (Eq. (3.12)) whose center manifold is controlled

$$w(x) = 2x^4\left(1 - x^2\right) - 1.5x^2$$

to effectively control the center manifold to lie on the curve $y = c_0x^2 = -x^2$.

Since $-1 = c_0 < -a = -0.5$, the unstable nonhyperbolic equilibrium point at the origin in the uncontrolled system is converted to an asymptotically stable equilibrium point. Figure 3.3b shows the behavior of the controlled system. One notes that though the statements in the previous section are true for "sufficiently small" initial conditions, the basin of attraction of the origin controlled in this manner is quite generous. More importantly, (a) asymptotic stability is guaranteed by the first three center manifold properties of the previous section without the need to appeal to the usual Lyapunov direct method that entails finding an appropriate Lyapunov function, and (b) the system has an explicitly described center manifold, and trajectories starting on or close to it (and from a generous enough region around the nonhyperbolic equilibrium point) will very closely track the trajectory given by:

$$x(t) = u(t),\ y = c_0x^2 = -x^2$$

where $\dot{u} = (c_0 + a)\,u^3 + u^5 = -0.5u^3 + u^5$. Evidently, any value of $c_0 < -a$ can be chosen depending on practical considerations that may arise when implementing the control, thereby controlling the center manifold.

Finally, there is no difficulty posed when $a + c = 0$ in order to ensure asymptotic stability of the origin as was previously encountered with the uncontrolled system (see Eq. (3.11)). If $a > 0$, we see from Eq. (3.14) that the system can be controlled to have a center manifold $y = -c_0x^2$, $c_0 > a$, so that the nonhyperbolic equilibrium point at the origin is asymptotically stable (in the vicinity of the origin). Similarly, when $a < 0$, the center manifold can be controlled to be $y = -c_0x^2$, $c_0 > 0$.

This leads us to the following result.

Result Given the system described by Eq. (3.1), and a C^1 function $y = h(x)$ from $\mathbb{R}^n \to \mathbb{R}^m$, the system:

$$\begin{aligned}\dot{x} &= Ax + f(x, y)\\ \dot{y} &= By + g(x, y) + w(x)\end{aligned} \tag{3.15}$$

The feedback control $w(x) \in C^2$, $w(0) = 0$, $Dw(0) = 0$ where:

$$w(x) = Dh(x)\left[Ax + f(x, h(x))\right] - Bh(x) - g(x, h(x)) \tag{3.16}$$

causes the system (3.15) to have the center manifold $y = h(x)$. The dynamics on this center manifold is described by the n-dimensional equation:

$$\dot{u} = Au + f(u, h(u)). \tag{3.17}$$

Proof: We simply apply the Existence property to the system described by Eq. (3.15). We note that the control is not full-state and is applied to the m-dimensional subsystem that is described by the y-equation in Eq. (3.15). □

Remark 3 The other three properties given in Sect. 3.1 above follow in a similar manner. The Asymptotic Behavior of Trajectories property assures us that for each trajectory of the full dynamical system (3.15) (provided it stays sufficiently close to the nonhyperbolic equilibrium point), there is a particular solution of the lower-dimensional system (3.17) on the center manifold that is approached exponentially fast.

The central idea behind the Result is to provide a control $w(x)$ so that system (3.15), which has a nonhyperbolic equilibrium point at the origin, has a suitable center manifold that attracts nearby orbits in a region around the origin and makes it (the origin) asymptotically stable.

We now use this methodology to the area of mechanics that deals with stability of the nonhyperbolic equilibrium point that arises at the origin when the angular velocity of a rigid body is controlled.

Consider the Euler equations for the rotation of a rigid body in the absence of external torques in the angular velocity space $(\omega_1, \omega_2, \omega_3)$, a problem related to satellite attitude control in deep space. The aim is to ensure that the system remains asymptotically stable for small perturbations around the origin. Assuming that the moments of inertia (MI) about the three principal axes are I_1, I_2, I_3, we obtain the equations:

$$\begin{aligned}\dot{\omega}_1 &= a\omega_2\omega_3 - \alpha_1\omega_1^3\\ \dot{\omega}_2 &= b\omega_3\omega_1 - \alpha_2\omega_2^3\\ \dot{\omega}_3 &= c\omega_1\omega_2 - \alpha_3\omega_3\end{aligned} \tag{3.18}$$

where:

$$a = \frac{(I_2 - I_3)}{I_1}, b = \frac{(I_3 - I_1)}{I_2}, \quad \text{and} \quad c = \frac{(I_1 - I_2)}{I_3}. \tag{3.19}$$

Our aim is to have the origin $(0, 0, 0)$ asymptotically stable. To achieve stability, we take the parameters α_1, α_2, $\alpha_3 > 0$. Thus, the coordinate ω_3 is provided linear negative feedback, while the coordinates ω_1 and ω_2 are provided negative cubic feedbacks. For small perturbations from the origin, the control torques resulting from the cubic feedback terms would be negligible compared to those resulting from the use of a linear term, resulting in reduced control costs, hence the motivation to use nonlinear cubic feedback for the evolution of ω_1 and ω_2. We consider different cases depending on the values of the three principal moments of inertia.

The central idea is to provide a suitable control $w(x)$ so that the controlled system has a suitable center manifold that attracts nearby orbits in a region around the nonhyperbolic origin and makes it (the origin) asymptotically stable.

Case 1 $I_1 < I_3 < I_2$. I_3 is the intermediate principal moment of inertia (MI); the linear feedback is provided to the coordinate that corresponds to this intermediate moment of inertia I_3. From Eq. (3.19) we see that a, $b > 0$ and $c < 0$. Note that the linear feedback is provided to the coordinate ω_3 which corresponds to the intermediate moment of inertia, I_3.

The system can be rewritten as:

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} a\omega_2\omega_3 - \alpha_1\omega_1^3 \\ b\omega_3\omega_1 - \alpha_2\omega_2^3 \\ c\omega_1\omega_2 \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} f_1(\omega_1, \omega_2, \omega_3) \\ f_2(\omega_1, \omega_2, \omega_3) \\ g(\omega_1, \omega_2, \omega_3) \end{bmatrix} \tag{3.20}$$

showing that the equilibrium point at the origin, $\omega_1 = \omega_2 = \omega_3 = 0$, is nonhyperbolic. From the nature of the feedback control used here, one might intuit that the equilibrium point is stable. By the Existence property in Sect. 3.1, the system has a center manifold, described by the equation $\omega_3 = h(\omega_1, \omega_2)$. The center manifold of the system can be obtained by using Eq. (3.4) which yields

$$\frac{\partial h}{\partial \omega_1}\left[a\omega_2 h - \alpha_1\omega_1^3\right] + \frac{\partial h}{\partial \omega_2}\left[a\omega_1 h - \alpha_2\omega_2^3\right] + \alpha_3 h - \omega_1\omega_2 = 0.$$

An approximate solution of this equation (Eq. (3.5)) assuming that

$$\tilde{h} = \frac{c}{\alpha_3}\omega_1\omega_2 + O\left(\|\omega\|^4\right) \tag{3.21}$$

gives the center manifold, according to the Approximation property given in Sect. 3.1, as

$$\omega_3 = \frac{c}{\alpha_3}\omega_1\omega_2 + O\left(\|\omega\|^4\right) \tag{3.22}$$

and on this manifold, by Eq. (3.2) the dynamical system evolves according to the relations

$$\begin{aligned}\dot{u}_1 &= \tfrac{ac}{\alpha_3}u_1u_2^2 - \alpha_1u_1^3 + O\left(\|u\|^6\right) = u_1\left(\gamma_1u_2^2 - \alpha_1u_1^2\right) + O\left(\|u\|^6\right)\\ \dot{u}_2 &= \tfrac{bc}{\alpha_3}u_1^2u_2 - \alpha_2u_2^3 + O\left(\|u\|^6\right) = u_2\left(\gamma_2u_1^2 - \alpha_2u_2^2\right) + O\left(\|u\|^6\right)\end{aligned} \tag{3.23}$$

where $\gamma_1 = \frac{ac}{\alpha_3} < 0$ and $\gamma_2 = \frac{bc}{\alpha_3} < 0$. Using the Lyapunov function $V(u_1, u_2) = \left(u_1^2 + u_2^2\right)/2$ we find that

$$\dot{V} = u_1\dot{u}_1 + u_2\dot{u}_2 = -\alpha_1u_1^4 - \alpha_2u_2^4 + (\gamma_1 + \gamma_2)\,u_1^2u_2^2 < 0$$

so that Eq. (3.23) is asymptotically stable at the origin. It should be noted that the truncated Eq. (3.23) has only one equilibrium point, except when $\gamma_1\gamma_2/(\alpha_1\alpha_2) = 1$, which is a situation that can be excluded from consideration because the feedback gains α_1, α_2 and α_3 can always be appropriately selected.

Instead of thinking of the local center manifold as given approximately by the relation $w_3 = \frac{c}{\alpha_3}\omega_1\omega_2 + O\left(\|\omega\|^4\right)$, we can start by controlling the system so that its center manifold is exactly $w_3 = c_0\omega_1\omega_2$. In fact, one can consider making the local center manifold of the controlled system to be:

$$\omega_3 = c_0\omega_1\omega_2 + c_1\omega_1\omega_2^3 = h(\omega_1, \omega_2) \tag{3.24}$$

in which the constant coefficients c_0 and c_1 will be chosen to ensure asymptotic stability of the nonhyperbolic equilibrium point at the origin, accompanied by a generous neighborhood around it that the equilibrium point attracts.

The controlled system can be written as:

$$\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -\alpha_3\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} + \begin{bmatrix}a\omega_2\omega_3 - \alpha_1\omega_1^3\\ b\omega_3\omega_1 - \alpha_2\omega_2^3\\ c\omega_1\omega_2 + w(\omega_1, \omega_2)\end{bmatrix} \tag{3.25}$$

and it is identical to that shown in Eq. (3.20), except for the addition of a nonlinear control torque in the last equation. Substituting for $h(\omega_1, \omega_2)$ the expression given in Eq. (3.16), we obtain explicitly:

$$w\left(\omega_1,\omega_2\right)=\omega_1\omega_2\Big(c_0^2b\omega_1^2+c_0^2a\omega_2^2+4c_0c_1b\omega_1^2\omega_2^2+2c_0c_1a\omega_2^4-c_0\alpha_1\omega_1^2-c_0\alpha_2\omega_2^2 \\ +c_0\alpha_3+3c_1^2b\omega_1^2\omega_2^4+c_1^2a\omega_2^6-c_1\alpha_1\omega_1^2\omega_2^2-3c_1\alpha_2\omega_2^4+c_1\alpha_3\omega_2^2-c\Big) \tag{3.26}$$

Using Eq. (3.24) in Eq. (3.17), we get:

$$\begin{aligned}\dot{u}_1 &= u_1\left(c_0au_2^2-\alpha_1u_1^2\right)+c_1au_1u_2^4\\ \dot{u}_2 &= u_2\left(c_0bu_1^2-\alpha_2u_2^2\right)+c_1bu_1^2u_2^3.\end{aligned} \tag{3.27}$$

The Stability property in Sect. 3.1 tells us that if Eq. (3.27) is stable at the origin, then the controlled system will also be stable at the origin. Consider the candidate Lyapunov function:

$$V\left(u_1,u_2\right)=\frac{1}{2}\left(u_1^2+u_2^2\right).$$

Then the time derivative of V along the orbit of the dynamical system (3.27) is simply:

$$\dot{V}=u_1\dot{u}_1+u_2\dot{u}_2=-\alpha_1u_1^4-\alpha_2u_2^4+c_0\left(a+b\right)u_1^2u_2^2+c_1\left(a+b\right)u_1^2u_2^4, \tag{3.28}$$

which is guaranteed to be negative definite for all c_0, $c_1 < 0$, since a, $b > 0$. Thus, by controlling the center manifold to satisfy Eq. (3.24), the nonhyperbolic equilibrium point of the controlled system (3.25) is guaranteed to be stable for values of c_0, $c_1 < 0$. Notice the absence of the parameter c in Eq. (3.28) indicating that the stability of the origin has been made independent of this parameter.

Numerical Example 3 Consider the control of the center manifold of the system described by Eq. (3.18) so that the center manifold is given exactly by $y = -\omega_1\omega_2$. Then the control torque required in Eq. (3.25) is

$$w\left(\omega_1,\omega_2\right)=-\omega_1\omega_2\left(-c_0^2b\omega_1^2-c_0^2a\omega_2^2+c_0\alpha_1\omega_1^2+c_0\alpha_2\omega_2^2-c_0\alpha_3+c\right).$$

On this center manifold, the trajectories are given by Eq. (3.27) with $c_0 = -1$, and $c_1 = 0$.

Figure 3.4a shows the behavior of system (3.25) in which the moments of inertia are $I_1 = 1$, $I_2 = 2.5$ and $I_3 = 2$ in consistent units, so that $a = 0.5$, $b = 0.4$, and $c = -0.75$. The parameters $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$. These parameter values for α_1, α_2, and α_3 will be used throughout for all the cases except in Figs. 3.7 and 3.8 that look at the influence of these parameters on the controlled phase portrait of the system. The center manifold is shown by the shaded surface, and the start of each trajectory is shown by a small sphere. Exponential convergence to the center

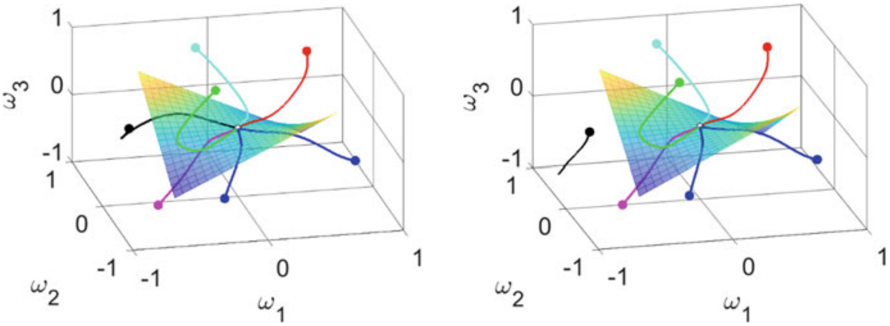


Fig. 3.4 (**a**) System (3.18) is controlled so that its center manifold is $\omega_3 = -\omega_1\omega_2$. (**b**) System (3.18) is controlled so that its center manifold is $\omega_3 = -\omega_1\omega_2 - 0.2\omega_1\omega_2^2$

manifold occurs, and asymptotic stability of the nonhyperbolic equilibrium point at the origin is seen. The representative trajectories are tangent to the center manifold at the origin and are exponentially attracted to the nonhyperbolic equilibrium point at the origin over a generous region of initial conditions around the origin. Note how all the trajectories are "guided" to lie on the local center manifold as they approach the equilibrium point.

Figure 3.4b shows the behavior of the system controlled to have a center manifold defined by $\omega_3 = -\omega_1\omega_2 - 0.2\omega_1\omega_2^3$ using the same initial conditions for the representative trajectories as in Fig. 3.4a. The other parameter values chosen are the same as above. While the trajectories remain, as expected, tangent at the origin to the center manifold, and the origin is asymptotically stable, the domain of attraction of the nonhyperbolic equilibrium point has been somewhat diminished as seen from the one trajectory that is not attracted any more to the origin.

Case 2

$I_2 < I_3 < I_1$. I_3 is again the intermediate moment of inertia (MI), and from Eq. (3.19) $a, b < 0$ and $a, b < 0$. To ensure that the nonhyperbolic equilibrium point at the origin is asymptotically stable, we need $\dot{V}$ given in Eq. (3.28) to be negative definite; hence, we control the system to achieve this objective, by controlling it to have a center manifold given by Eq. (3.24) with $c_0, c_1 > 0$; this ensures that $\dot{V}$ is negative definite. A simple way to see that a control manifold can be used to make the origin stable is simply to rename the second principal axis as the first principal axis in Case 1 and vice versa.

Case 3

$I_1 < I_2 < I_3$. I_2 is now the intermediate principal moment of inertia (MI), and I_3 is the largest principal MI. Now, $a, c < 0$ and $b > 0$. To ensure asymptotic stability of the origin when $(a + b) > 0$, from Eq. (3.28) we see that we can control the center manifold to be that given by Eq. (3.24) with $c_0, c_1 < 0$. When $(a + b) < 0$, from Eq. (3.28) we obviously require $c_0, c_1 > 0$.

Numerical Example 4 Consider the rigid body with $I_1 = 1, I_2 = 2, I_3 = 2.5$ so that $a = -0.5$, $b = 0.75$, and $c = -0.4$. Here $a + b > 0$ and we control the local

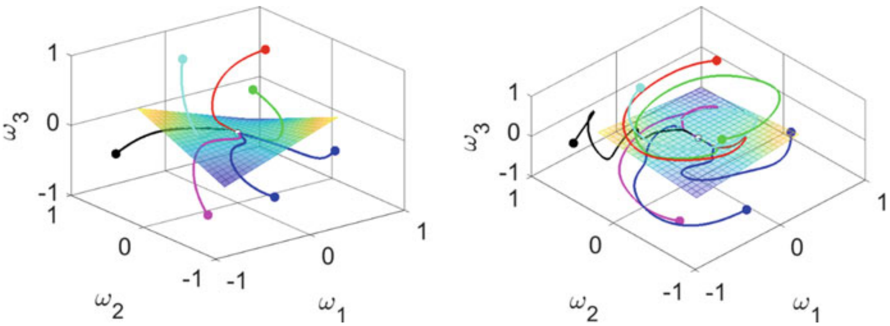


Fig. 3.5 (**a**) Stability of origin when $I_1 < I_2 < I_3$ and $a + b > 0$. (**b**) Stability of origin when $I_1 < I_2 < I_3$ and $a + b < 0$

center manifold of the system so that it is given by $y = -\omega_1\omega_2$ (Eq. (3.24) with $c_0 = -1$ and $c_1 = 0$). The parameters $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$. The phase portrait of the controlled system is shown in Fig. 3.5a. As seen, the origin is asymptotically stable with a generous region in phase space around it that it attracts.

With $I_1 = 1$, $I_2 = 2$, $I_3 = 6$, we have $a = -4$, $b = 2.5$, and $c = -1/6$ so that $(a + b) < 0$. By controlling the center manifold to be as given in Eq. (3.24) with $c_0 = 1.2$, and $c_1 = 0.2$ the nonhyperbolic origin is made asymptotically stable as seen in Fig. 3.5b. Asymptotic stability and trajectories tangential to the center manifold are again seen.

Using similar reasoning, the center manifold when $I_2 < I_1 < I_3$ can also be obtained. Now $a < 0$ and $b, c > 0$. The parameter c does not influence our choice of the parameters that describe the desired center manifold, only the parameters a and b matter is assuring asymptotic stability of the origin (Eq. (3.28)). Hence, this is the same as Case 3 discussed earlier, and stability can be assured by a proper choice of a center manifold.

Case 4

$I_3 < I_1 < I_2$. Here I_1 is the intermediate principal MI and hence the parameters $a > 0$ and $b, c < 0$, which is the same as Case 3, which has already been discussed. When $(a + b) > 0$, stability of the equilibrium point at the origin is assured by controlling the system to have Eq. (3.24) be its center manifold with $c_0, c_1 < 0$.

Numerical Example 5 Considering the system $I_1 = 2$, $I_2 = 3.5$, $I_3 = 1$, with the parameters $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$ so that $a = 1.25$, $b = -0.286$, $c = -1.5$. The controlled local manifold is taken to be Eq. (3.24) with $c_0 = -1.2$, and $c_1 = -0.2$. The phase portrait is shown on Fig. 3.6. As before, the spheres show the initial conditions from which the representative trajectories start. Asymptotic convergence to the stable nonhyperbolic equilibrium point at the origin is seen from a sizable region of phase space. All the trajectories are guided by the control to lie on the local center manifold $\omega_3 = -1.2\omega_1\omega_2 - 0.2\omega_1\omega_2^3$ as they approach the equilibrium point.

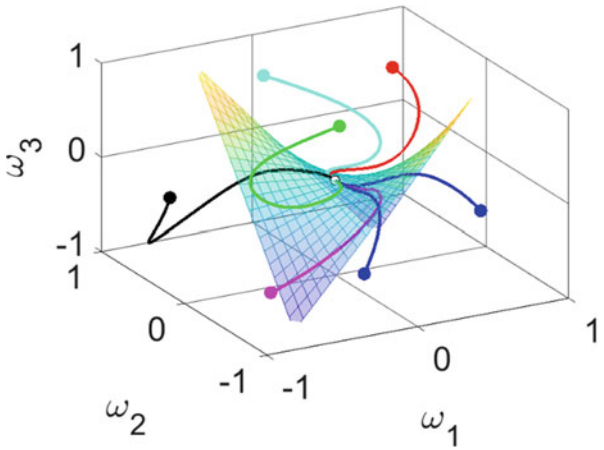


Fig. 3.6 $I_1 < I_2 < I_3$ and $a + b > 0$

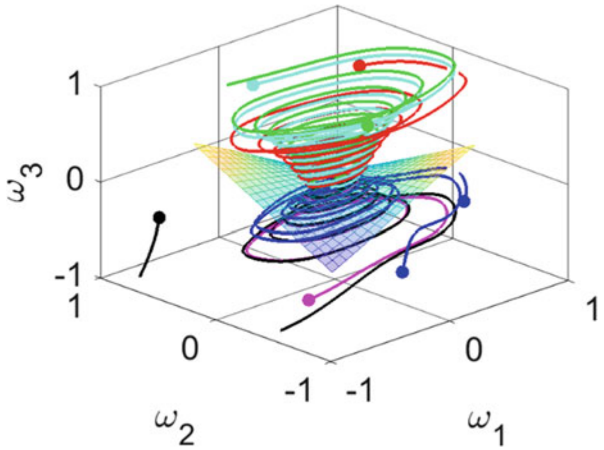
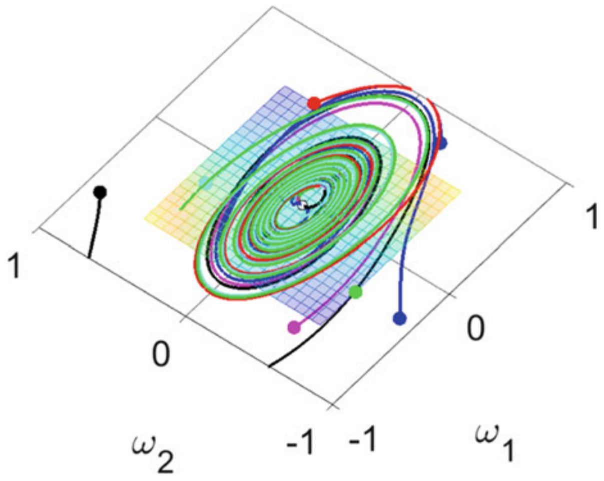


Fig. 3.7 $\alpha_1 = 0, \alpha_2 = 0,$ and $\alpha_3 = 0.01$

Lastly, we consider the effect of the system's parameters on the efficacy of the control. With values of I_1, I_2, and I_3 the same as above we consider a system with greatly reduced feedback gains (see Eq. (3.18)) of $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0.01$. Thus, there is no feedback to the first two equations, and the feedback to the third equation is drastically reduced. The nonlinear system is controlled in the vicinity of its nonhyperbolic fixed point by controlling the local center manifold to be $\omega_3 = -1.2\omega_1\omega_2 - 0.2\omega_1\omega_2^3$. Figure 3.7 shows the phase portrait of the system and the center manifold, and Fig. 3.8 shows the projection of the phase portrait on the $\omega_1 - \omega_2$ plane. As seen, trajectories starting from the same points as those in Fig. 3.6 again approach the equilibrium point along the center manifold. The trajectory starting at the leftmost point in Fig. 3.7 appears to go out a distance beyond the limits of the plot shown and returns to the center manifold.

Fig. 3.8 $\alpha_1 = 0, \alpha_2 = 0,$ and $\alpha_3 = 0.01$



3.3 Conclusions

The analysis of nonlinear systems with hyperbolic equilibrium points is greatly facilitated by the Grobman-Hartman (GH) result that states that in the vicinity of such an equilibrium point, the linearized system is topologically equivalent to the nonlinear system. This result makes possible the control of such systems in a straightforward and simple manner. However, the presence of nonhyperbolic equilibrium points makes the GH result inapplicable, and this often poses challenges in our understanding, our analysis, and our ability to control such systems when operating in the vicinity of such nonhyperbolic equilibrium points. This chapter explores a new approach to the control of such systems using results from the theory of center manifolds.

The central idea is to put in place a control on an easily distinguishable part (subsystem) of the dynamical system that results in a desirable local center manifold which ensures asymptotic stability of the nonhyperbolic equilibrium point. The desired center manifold would depend, in general, on practical considerations such as actuator requirements, the region of attraction required around the nonhyperbolic equilibrium point in phase space, etc. A straightforward methodology for doing this is developed. The system is controlled to have a desired local center manifold upon which the evolving dynamics is assured to be asymptotically stable.

The nonlinear dynamical system is divided into two parts. The first part constitutes the subsystem that has the nonhyperbolic equilibrium point, and the remaining hyperbolic subsystem constitutes the second part. As seen, the control effort needs to be applied only to this remaining subsystem. Therefore, the methodology has the advantage of not requiring full-state control while guaranteeing asymptotic stability.

While one could use Lyapunov's direct method in some instances as done here, a disadvantage of the approach presented here is the difficulty in exactly finding the region of attraction of the asymptotically stable nonhyperbolic equilibrium point after a desirable local center manifold is chosen. As illustrated here, the size of this region of attraction in phase space depends on the structure of the nonlinear system, on the parameters of the system, and on the choice of the desired local center manifold.

Though it has been assumed here that the second remaining subsystem, which is hyperbolic, is stable, extensions of the approach where it could contain both stable and unstable hyperbolic equilibrium points can be easily made by using control methods standardly used when dealing with hyperbolic equilibrium points.

The results obtained here appear to provide a new additional quiver in our arsenal of methods to bring about control of nonlinear dynamical systems, the focus in this chapter being on those systems that have nonhyperbolic equilibrium points in whose vicinity: (i) the nonlinear dynamics is usually more complex, and (ii) linearization methods do not work. As often done when dealing with the theory of center manifolds, examples are provided, one of which has considerable relevance to rigid body dynamics and deals with the control of the angular velocity of a rotating rigid body. The advantages of using the approach, as illustrated by these examples, are its simplicity, efficacy, and ease of use.

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